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Quantum Neural Network

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In the neural network theory content-addressable memories are defined by patterns that are attractors of the dynamical rule of the system. This paper develops a quantum neural network starting from a classical neural network Hamiltonian and using a Schrödinger-like equation. It then shows that such a system exhibits probabilistic memory storage characteristics analogous to those of the dynamical attractors of classical systems.

1. INTRODUCTION

The field of quantum mechanical computing is as yet in its infancy with only a few theoretical models and almost no physical hardware to display. Despite this, it is the subject of considerable interest due to the necessity of understanding computational limits brought on by quantum effects and the ever-decreasing size of electronic componentry (Landauer, 1982; Benioff, 1986; Feynman, 1985; Margolus, 1990). Additionally, there are perhaps lessons to be learned from both a physical and a computational standpoint by examining limiting cases of microscopic computing (Feynman, 1982; Margolus, 1986).

In its most general form, computing consists in manipulating information in some physically controlled manner. If it is performed by a system obeying the laws of classical physics, such a computation consists in a dynamical evolution from an initial to some final, observable state. In von Neumann computing this is accomplished through the use of serial logic gates consisting of binary elements following a set of predetermined instructions (an algorithm).

Feynman (1985) produced a theoretical quantum model that could, in principle, reproduce serial computation. Feynman noticed that the truth tables

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of all of the basic logical functions could be reproduced by quantum mechanical raising and lowering operators acting on up/down spin sites. In particular, an operator constructed from the required combination of raising and lowering operators (and their adjoints) was found to be Hermitian, permitting its use as the Hamiltonian in the Schrödinger equation.

The system consisted of a chain of spin sites connected by logic gates described by the Hermitian operator. In detail, these operators ensured that only one site at a time was operated on in sequence, changing the up/down states of the spin sites according to the rules the operators defined. Although no suggestion was made as to how to physically implement the site connections, at the same time there appeared to be no fundamental reason preventing their implementation. The system operated at a regular pace, and produced a reliable simulation of classical computation by confining quantum uncertainty exclusively to the time domain. In principle, this was a workable serial computer developed within the framework of the quantum formalism, evolving from an initial state to some measurable final state according to the rules of some algorithm.

Margolus (1990) extended Feynman's model to one that worked as a two-dimensional cellular automaton with nearest neighbor updating, showing in the process that the original was actually a specific case of this more general model. Thus was provided a formal description of parallel quantum computation to complement that of serial computation.

Since both nearest-neighbor automata and neural networks are members of the cellular automata family, Margolus' model naturally leads the curious to consider the possibility of quantum neural networks. Gerzon (1990), for instance, proposed a physical implementation of a rudimentary network. In addition, the neurobiologist Bray (1995) noted that while existing neural network models are only rough approximations of biological networks, they are actually quite good descriptions of protein signaling within cells. Remembering that proteins are themselves small enough to require consideration of quantum effects reveals the possibility of biological quantum computing.

Even though this remains for the moment conjecture, it is possible in the meantime to model systems of automata on computers. Specifically, it was shown by Kostin (1993) that cellular automata may be used to model both the Schrödinger and Dirac equations. This affords the possibility of at least comparing theory with model experiments until contact with actual experiment is made.

In this spirit, this paper develops a quantum neural network (QNN) model. This model is then examined to see if it possesses the most fundamental characteristic of classical neural networks—memory storage in attractors. It deviates, however, from Feynman's approach of creating from scratch a Hamiltonian that mimics classical computation when employed in the Schröd-

inger equation. Instead, the already existing classical Hamiltonian for continuous-valued neural networks introduced by Ramacher (1993) is used to derive a Schrödinger-like equation. Eschewing the need for exact mimicry of classical computation, specific examples of this system are instead compared to *analogous* classical examples to see if such systems exhibit *analogous* memory storage capabilities. As well, along the way one eye is kept peeled for results of general physical interest.

Specifically, the classical Hamiltonian used is introduced in Section 2, followed by a stability analysis of examples of such a system in Section 3. Section 4 then derives the general form of the quantum neural network, followed in Section 5 by solutions to the resulting Schrödinger-like equation for cases equivalent to those in Section 3.

2. THE CLASSICAL HAMILTONIAN

In his paper, Ramacher showed that nontrivial nets of continuous-valued neurons can be conveniently accommodated in a Hamiltonian framework. Since activation and weight dynamics of the nets are derived from a single partial differential equation of the Hamilton–Jacobi type, the approach lends itself to interesting applications. Of special interest is the case where the neural activities alone are dynamical variables and the connection weights are fixed parameters. This restricts learning to "off-line" methods, but the classical Hamiltonian used is particularly simple and reproduces the structure of backpropagation. For the sake of completeness, we summarize below Ramacher's approach.

In a neural network of N units, the activity state of a neuron i at time t is denoted by $y_i(t)$, with the state of the entire system defined by the N-dimensional vector $\mathbf{y}(t)$. In its most general form, this state is determined by the site-to-site connection strengths w_{ij}, the external inputs Y_i, and the individual thresholds Θ_i of the neurons. Among the external inputs may be included learning inputs or cost functions used to implement various learning schemes.

For a given set of connection weights defined by the weight vector $\mathbf{w}$, the external inputs and thresholds form the boundary conditions for which the equations of motion of the system may be integrated. This determines the trajectory $J(t, \mathbf{y}(t))$ of some variable of interest in the activity space. Compiling the possible trajectories resulting from all of the possible boundary conditions forms a surface $J(t, \mathbf{y}; \mathbf{w})$ for a particular set of weights (listing variables after the semicolon is understood here to denote parameters). These surfaces contain information about the dynamics of the network, and satisfy

in general a partial differential equation. The latter is restricted heuristically to be of the Hamilton–Jacobi form

$$\frac{\partial J}{\partial t} + h(t, \mathbf{y}, \boldsymbol{\Delta}; \mathbf{w}) = 0 \tag{1}$$

where

$$\Delta_i = \frac{\partial J}{\partial y_i} \tag{2}$$

Since a single surface is defined by a single set of connection weights, it thus corresponds to a single possible "lesson" or "memory." The entire family of possible surfaces corresponds to the complete possible set of connection weights.

Here, Ramacher chooses for equation (1) a Hamiltonian that is linear in the conjugate variable $\boldsymbol{\Delta}$ and has the general form

$$h = \sum_{j=1}^{N} \Delta_j F_j(t, \mathbf{y}; \mathbf{w}) + \mathbf{E}(t, \mathbf{y}; \mathbf{w}) \tag{3}$$

where $\mathbf{E}$ is included as the error or cost function for possible learning schemes. Equation (1) leads to the characteristic equations

$$\frac{dy_i}{dt} = F_i \tag{4}$$

$$\frac{d\Delta_i}{dt} = -\sum_{j=1}^{N} \Delta_j \frac{\partial F_j}{\partial y_i} - \frac{\partial E}{\partial y_i} \tag{5}$$

$$\frac{dJ}{dt} = -h + \sum_j \Delta_j F_j \tag{6}$$

Equations (4) describe the activity dynamics of the neurons. They are reobtained as a characteristic equation because of the particularly simple form of Equation (3). Equation (5) leads to a time-dependent generalization of the delta of backpropagation. We follow Ramacher in considering the special case of the continuous-valued Hopfield model, where the characteristic equations for y_i and Δ_i become

$$\frac{dy_i}{dt} = \frac{1}{\lambda}\left(-y_i + f_i(Y_i + \Theta_i + \sum_j w_{ij} y_j)\right) \tag{7}$$

$$\frac{d\Delta_i}{dt} = \frac{1}{\lambda}\left(\Delta_i - \sum_j \Delta_j f_j'(Y_i + \Theta_i + \sum_j w_{ij} y_j) w_{ji}\right) - \frac{\partial E}{\partial y_i} \tag{8}$$

Here f_i is some squashing function and λ is the general time constant for the neurons. Ramacher studied these equations numerically for their effectiveness in time-dependent recurrent backpropagation schemes. In contrast, this paper focuses on analytical stability analysis of the equilibria of simple nonlearning Hopfield networks.

3. DYNAMICAL STABILITY ANALYSIS

The purpose of this paper is to see whether the QNN model possesses memory storage characteristics analogous to those of an equivalent classical system. The problem is not trivial, in view of the probabilistic nature of quantum mechanical predictions. A memory in a classical neural network is stored as the location in the activity space of the stability point of a dynamical attractor. It is understood that a guarantee of dynamical stability is a prerequisite for a network to be useful as a memory storage device. Further, it has been shown (Hopfield, 1982, 1984) that this is provided in both the discrete- and continuous-valued Hopfield networks through the use of an energy (or Lyapunov) function. This guarantee only holds, however, for networks with symmetric ($w_{ij} = w_{ji}$) connection weights. The introduction of a full range of connection weights requires use of dynamical systems theory. To avoid unnecessary complication, stability analysis of the dynamics of the continuous-valued Hopfield network is undertaken for the simplest case. That is, a noiseless 2-neuron system (X–Y model) with common gain function and no external input. Even as such, approximations must be made; these are repeated when the quantum system is analyzed and compared.

Referring to equation (4), the dynamics of the system is given by

$$\frac{dy_i}{dt} = -y_i + g\left(\sum_{j\neq i}^{2} w_{ij}y_j\right) \tag{9}$$

where the time constant τ_i has been set to equal 1 and self-connections w_{ii} to 0. In the interest of further simplicity, the squashing function $g(h)$ to be chosen is the Glauber function. Written in full, the differential equations are

$$F_i = \frac{dy_i}{dt} = -y_i + \frac{1}{1 + e^{-w_{ij}y_j}} \tag{10}$$

where if $i = 1$, then $j = 2$ and vice versa. This is understood from here on. Equations (10) confine $y_i(t)$ to an activity space I where $0 \leq y_i \leq 1$.

Next, the locations of attractors in activity space are defined by the locations $y_i^* = (y_1^*, y_2^*)$ of the equilibrium points where $dy/dt = 0$. These give the transcendental equations

$$y_i^* = \frac{1}{1 + e^{-w_{ij}y_j^*}} \tag{11}$$

which may only be solved numerically. The curves defined in equations (11) intersect only once, always within the $I = [0, 1]$ activity space. Therefore the system has exactly one equilibrium point for a particular combination of connection weights.

A map (see Tables (I) and (II)) of the equilibrium points has been calculated numerically to illustrate how the location of the equilibrium point moves as a function of changes in the connection weights. The main result to be noticed is that the location of the equilibrium point moves away from the center of the activity space as the magnitude of the connection weights increases. This effect is very nearly linear at the center of the weight space, and becomes less pronounced, eventually becoming negligible, as the boundaries of I are approached. Reintroduction of the sharpness factor 2β merely shifts the locations on the map. In other words, dividing the connection weights by 2β restores the original map.

With this map in hand, the equilibrium points may now be categorized as either stable (attractors) or unstable (repellors), determining which ones may be useful for memory storage. The analysis used to determine which equilibrium points are actually stable attractors is conducted using the local linearization of the Jacobian matrix $A = \partial F_i/\partial y_j|_{y_i^*}$ of the equations of motion in conjunction with the Hartman–Grobman linearization theorem (Tu, 1994).

The linearization of equations (10) can be defined using the new variables $\xi = (y_1 - y_1^*)$ and $\mu = (y_2 - y_2^*)$ as

$$\begin{pmatrix} \dot{\xi} \\ \dot{\mu} \end{pmatrix} = A \begin{pmatrix} \xi \\ \mu \end{pmatrix} + \begin{pmatrix} \phi_1(\xi, \mu) \\ \phi_2(\xi, \mu) \end{pmatrix} \tag{12}$$

which is merely a Taylor-series expansion (to first order) about y_i^*. The ϕ_i terms are variations that may be disregarded as small in the neighborhood of the point.

Table I. Map of Stability Points for Full Range of Connection Weights.

w_{21}	y_1/y_2							
6	.552/.035	.616/.078	.749/.183	.953/.500	.995/.880	.996/.952	.997/.997	
4	.538/.038	.586/.087	.692/.200	.887/.500	.971/.875	.981/.981	.982/.997	
2	.521/.042	.551/.102	.612/.227	.737/.500	.844/.844	.875/.971	.880/.995	
0	.500/.047	.500/.120	.500/.276	.500/.500	.500/.737	.500/.881	.500/.953	
−2	.472/.056	.423/.156	.337/.337	.269/.500	.227/.612	.200/.672	.183/.749	
−4	.430/.070	.260/.260	.156/.423	.120/.500	.102/.551	.087/.586	.078/.616	
−6	.216/.216	.070/.430	.056/.472	.045/.511	.042/.521	.038/.538	.035/.552	
	−6	−4	−2	0	2	4	6	w_{12}

Table II. Map of Stability Points For $-1 \leq w_{ij} \leq 1^a$.

w_{21}	y_1/v_2										
1.0	.59/.36					.38/.50				.65/.65	.66/.66
0.8									.62/.59	.62/.62	.65/.65
0.6			.56/.42					.58/.58	.59/.59	.59/.62	
0.4								.58/.58			
0.2					.52/.47	.52/.50	.57/.57				
0	.50/.38				.50/.47	.50/.50	.50/.52				.50/.62
−.2					.48/.48	.48/.50	.47/.52				
−.4		.46/.43	.46/.46				.45/.54				
−.6			.44/.44						.42/.56		
−.8											
−1	.40/.40					.62/.50					.36/.59
	−1.0	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8	1.0 w_{12}

[a] Rounded to two decimal places due to space constraints.

Putting the Jacobian A into equation (12), and remembering that $\dot{\xi} = \dot{y}_i$ and $\dot{\mu} = \dot{y}_2$, we obtain the linearized equations of motion

$$\dot{y}_i = -y_i + y_i^* + \frac{w_{ij}e^{-w_{ij}y_j^*}}{(1 + e^{-w_{ij}y_j^*})^2}(y_j - y_j^*) + \phi_i \tag{13}$$

The key to analyzing the stability of the equilibrium points is a set of rules based on the form of the eigenvalues of the local Jacobian matrix (Tu, 1994). For present purposes, these rules require that for a stable attractor, the real part of each eigenvalue be negative, or else the equilibrium is unstable.

If the Jacobian matrix for the 2-neuron system is written in the form

$$A = \begin{pmatrix} -1 & \Gamma_2 \\ \Gamma_1 & -1 \end{pmatrix} \tag{14}$$

then the eigenvalues are

$$\lambda_{1,2} = -1 \pm \sqrt{\Gamma_1\Gamma_2} \tag{15}$$

From the guidelines listed above, it is apparent that the system under consideration has a stable attractor for all cases, unless $\Gamma_1\Gamma_2 > 1$. Evaluating the eigenvalues of the map of equilibrium points (Tables I and II), the result is that only systems with highly inhibitory weights (both w_{ij}'s < -6 or so) violate the stability condition. It is important to notice that since two-dimensional systems cannot exhibit chaos and trajectories cannot cross, it is reasonable to presume that, for this case, local stability implies global stability. Of importance as well is that almost the entire activity space is available for memory storage.

A further approximation may be introduced, using the original equilibrium locations shown in Table I. If in equations (10), the y_i terms in the $g(h)$ are replaced by y_i^*, the equilibrium point remains the same (memories have the same location), but the Jacobian matrix is now simply

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{16}$$

for all cases. The eigenvalues are now simply $\lambda_{1,2} = -1$, which guarantees stability, and a straight-line trajectory to the equilibrium point. This is therefore a valid approximation only of systems originally exhibiting a stable equilibrium.

With this information now in hand, the stage is set to proceed with the development and analysis of the QNN model.

4. CONSTRUCTING THE QUANTUM NEURAL NETWORK

It is assumed here that the usual framework of quantum mechanics remains valid even in the nonlinear case, as supported by available experimental evidence. A quantum Hamiltonian is therefore assigned to the known classical process by means of the usual formal replacements $\partial/\partial t \rightarrow i\alpha\ \partial/\partial t$, $\Delta_i \rightarrow -i\alpha\ \partial/\partial y_i$ (where α plays the role of Planck's constant in the activity space) in equations (1)–(3), and by writing the product $\Delta_j F_j$ as $1/2\ (\Delta_j F_j + F_j \Delta_j)$ (Schiff, 1968). One obtains the Schrödinger-like equation

$$i\alpha \frac{\partial \Psi}{\partial t} = \left\{ -\frac{i\alpha}{2} \sum_j \left(F_j \frac{\partial}{\partial y_j} + \frac{\partial F_j}{\partial y_j} \right) - \frac{i\alpha}{2} \sum_j F_j \frac{\partial}{\partial y_j} + E \right\} \Psi \tag{17}$$

where $\Psi = \Psi(t, \mathbf{y})$ is the wave function of the quantum neural net. By using the commutation relations

$$\frac{\partial}{\partial y_j} F_i(\mathbf{y}) - F_i(\mathbf{y}) \frac{\partial}{y_j} = \frac{\partial F_i}{\partial y_j}$$

one further obtains

$$i\alpha \frac{\partial \Psi}{\partial t} = \left(\mathbf{F} \cdot (-i\alpha \nabla_{\mathbf{y}}) - \frac{i\alpha}{2} \nabla_{\mathbf{y}} \cdot \mathbf{F} + E \right) \Psi \tag{18}$$

These equations define a general quantum mechanical network with activity dynamics, potential, external inputs, and thresholds. The activities are now Hermitian operators producing a "wave function" in some Hilbert space. The time derivative of these activity operators is thus

$$\frac{dy_j}{dt} = \frac{i}{\alpha} [H, y_j] + \frac{\partial y_i}{\partial t} \tag{19}$$

Using the Hamiltonian of equation (18), this commutator is

$$[H, y_j] = -i\alpha F_j \tag{20}$$

Therefore, presuming the neural activities to be time-independent, the "velocity" of the activity operators is simply

$$\frac{dy_j}{dt} = F_j \tag{21}$$

This is exactly the same as for the variables of the classical system. The physical content of equation (18) obviously exceeds that of equations (1)–(3). This can be seen by splitting equation (18) into its real and imaginary parts. By writing $\Psi(t, \mathbf{y}) = A(t, \mathbf{y}) \exp(-i\alpha J(t, \mathbf{y}))$, where A and J are real functions,

and substituting into equation (18), one obtains equation (1) from the real part and

$$\frac{\partial A}{\partial t} + \sum_j F_j \frac{\partial A}{\partial y_j} + \frac{1}{2} \sum_j \frac{\partial F_j}{\partial y_j} A = 0 \tag{22}$$

from the imaginary part. By multiplying equation (22) by A, one arrives at the equation

$$\frac{\partial A^2}{\partial t} + \sum_j \frac{\partial}{\partial y_j} (A^2 F_j) = 0 \tag{23}$$

which, because of equation (21) and the relation $A^2 = |\Psi|^2$, expresses the conservation of probability of finding the system in a bounded region of activity space. The probability interpretation of Ψ is possible only if equation (23) is satisfied (Schiff, 1968). This justifies *a posteriori* the quantization procedure followed. It must now be determined whether the probability distribution resulting from equation (18) is in some way capable of storing a memory in a manner analogous to the classical model.

5. WAVE FUNCTION FOR THE QNN

This section provides approximate solutions to equation (18) for a 2-neuron system equivalent to the X–Y models of Section 3. These are for the cases of symmetric weights of arbitrary magnitude, and of small-magnitude nonsymmetric weights, respectively. The form of the resulting probability distributions is then examined to determine how observable memories may be stored in such systems. These solutions may be extended to N-neuron systems without alteration in principle. Specifically, Section 5.1 uses the standard linearization of Section 3 to illustrate the similarity between the classical stability criteria and the requirement for probabilistic memory storage. The "small-weights" approximation of Section 3 is then employed in Section 5.2 to provide a simpler wave function, which is subsequently normalized in Section 5.3.

In the classical X–Y models, neither E nor $\mathbf{F}$ in equation (18) are explicit functions of time, therefore the spatial wave function $\Psi(\mathbf{y})$ obeys the energy eigenvalue equation

$$H\psi = \epsilon\psi \tag{24}$$

where ϵ plays the role of an energy, and the complete solution is presumed to be

$$\Psi(\mathbf{y}, t) = \psi(\mathbf{y}) \exp\left(-\frac{i\epsilon t}{\alpha}\right) \tag{25}$$

Equation (24) may then be written as

$$\left(-i\alpha\mathbf{F}\cdot\nabla_{\mathbf{y}}-\frac{i\alpha}{2}\nabla_{\mathbf{y}}\cdot\mathbf{F}+E\right)\psi=\epsilon\psi \tag{26}$$

by using equation (18).

To render this equivalent to the classical X–Y model, the potential E is set to 0 and F_i is again equation (10). It is easily checked that this Hamiltonian is Hermitian. Equation (26) becomes

$$-i\alpha\left(F_1\frac{\partial\psi}{\partial y_1}+F_2\frac{\partial\psi}{\partial y_2}\right)=\left[\epsilon+\frac{i\alpha}{2}\left(\frac{\partial F_1}{\partial y_1}+\frac{\partial F_2}{2y_2}\right)\right]\psi \tag{27}$$

Evaluation of the derivatives on the right-hand side leaves simply

$$F_1\frac{\partial\psi}{\partial y_1}+F_2\frac{\partial\psi}{\partial y_2}=\gamma\psi \tag{28}$$

where $\gamma = 1 + i\epsilon/\alpha$.

Here Lagrange's method of characteristics (Duffy, 1986) may be used to reduce equation (28) to the system of ordinary differential equations

$$\frac{dy_1}{F_1}=\frac{dy_2}{F_2}=\frac{d\psi}{\gamma\psi} \tag{29}$$

These equations, with F_i given by equation (27), do not appear to be integrable and require approximations for their solution.

5.1 Symmetric Connection Weights

To begin, the first of equations (29) in parametric form yields equations (10) again. It may also be written as

$$\frac{dy_1}{dy_2}=\frac{-y_1+1/[1+\exp(-w_{21}y_2)]}{-y_2+1/[1+\exp(-w_{21}y_1)]} \tag{30}$$

and the linearization approximation used in equation (12) for the dynamical stability analysis of the classical system again applies. Using the definition [equation (14)] of Γ_i and $\Delta y_i = y_i - y_i^*$, this can be written compactly as

$$F_i=-\Delta y_i+\Gamma_i\Delta y_j \tag{31}$$

Equation (30) is then

$$\frac{dy}{-\Delta y_1+\Gamma_1\Delta y_2}=\frac{dy_2}{-\Delta y_2+\Gamma_2\Delta y_1} \tag{32}$$

with the approximation presumed as before to be most valid in the neighborhood of the singularity. Likewise, the activity operators remain confined to I. Equation (32) may be integrated exactly only if $\Gamma_1 = \Gamma_2 \equiv \Gamma$. This is regrettable, but merely returns us to the old restriction of symmetric weights. Using multipliers, we find that equation (32) gives

$$\frac{d(\Delta y_1 + \Delta y_2)}{\Delta y_1 + \Delta y_2} = \frac{d(\Delta y_1 - \Delta y_2)}{\Delta y_1 - \Delta y_2} \tag{33}$$

or, after integration,

$$C_1 = \left|\frac{\Delta y_1 + \Delta y_2}{\Delta y_1 - \Delta y_2}\right| \tag{34}$$

Next, equating the right-hand term of equation (29) with the left-hand term in equation (33) gives

$$\frac{d(\Delta y_1 + \Delta y_2)}{\Delta y_1 + \Delta y_2} = \frac{d\psi}{\gamma\psi} \tag{35}$$

and upon integration

$$C_2 = \frac{\psi}{|\Delta y_1 + \Delta y_2|^{\gamma/(\Gamma-1)}} \tag{36}$$

The solution therefore consists of an arbitrary function of the form

$$\Phi(C_1, C_2) = 0 \tag{37}$$

The specific solution to the problem is now found using the input location of an initial wave function described in terms of a reference curve (Duffy, 1986). In this case it is presumed to take the form of a Gaussian centered on the point (y_1°, y_2°):

$$\psi^\circ(y_1, y_2) = A \exp\left\{-\frac{1}{4\sigma^2}[(y_1 - y_1^\circ)^2 + (y_2 - y_2^\circ)^2]\right\} \tag{38}$$

where A is the amplitude, and $1/(4\sigma^2)$ is the "sharpness" of the distribution, presumed to be chosen sharp enough that $\psi^\circ \sim 0$ at the boundaries of I. This is an attempt to mimic the initial starting point for classical trajectories in a quantum system. The base curve is chosen to be

$$y_1 = y_1^\circ \tag{39}$$

and so, along this curve equation (34) is

$$C_1 = \left|\frac{\Delta y_1^\circ + \Delta y_2}{\Delta y_1^\circ - \Delta y_2}\right| \tag{40}$$

where $\Delta y_1^\circ = y_1^\circ - y_1^*$. Solving for y_2 gives

$$y_2 = \pm \Delta y_1^\circ \left(\frac{C_1 + 1}{C_1 - 1} \right) + y_2^* \tag{41}$$

Here '+' corresponds to when the top and bottom arguments of equation (40) have the same sign, and the '−' is used for the case of opposite signs. This convention is to be understood from here on.

Together with equations (40) and (41), equation (38) becomes

$$\psi^\circ = A \exp\left\{ -\frac{1}{4\sigma^2} \left[\pm \Delta y_1^\circ \left(\frac{C_1 + 1}{C_1 - 1} \right) - \Delta y_2^\circ \right]^2 \right\} \tag{42}$$

Next, remembering Equation (36), we know

$$\psi = \psi^\circ \frac{|\Delta y_1^\circ + \Delta y_2|^{-\gamma/(\Gamma-1)}}{|\Delta y_1 + \Delta y_2|^{-\gamma/(\Gamma-1)}} \tag{43}$$

Together with equation (42), this gives

$$\psi = A \exp\left\{ -\frac{1}{4\sigma^2} \left[\pm \Delta y_1^\circ \left(\frac{C_1 + 1}{C_1 - 1} \right) - \Delta y_2^\circ \right]^2 \right\} \frac{|\Delta y_1^\circ + \Delta y_2|^{-\gamma/(\Gamma-1)}}{|\Delta y_1 + \Delta y_2|^{-\gamma/(\Gamma-1)}} \tag{44}$$

Finally, to provide a function valid for the entire activity space, the constant C_1 must be replaced with its original definition [equation (34)]. The result is

$$\psi = A \exp\left[-\frac{1}{4\sigma^2} \left(\pm \Delta y_1^\circ \frac{|(\Delta y_1 + \Delta y_2)/(\Delta y_1 - \Delta y_2)| + 1}{|(\Delta y_1 + \Delta y_2)/(\Delta y_1 - \Delta y_2)| - 1} - \Delta y_2^\circ \right)^2 \right] \times \frac{|\Delta y_1^\circ + \Delta y_2|^{\gamma/(\Gamma-1)}}{|\Delta y_1 + \Delta y_2|^{-\gamma/(\Gamma-1)}} \tag{45}$$

and the probability density becomes

$$|\Psi|^2 = \psi\psi^* = A^2 \exp\left[\frac{-1}{2\sigma^2} \left(\pm \Delta y_1^\circ \frac{|(\Delta y_1 + \Delta y_2)/(\Delta y_1 - \Delta y_2)| + 1}{|(\Delta y_1 + \Delta y_2)/(\Delta y_1 - \Delta y_2)| - 1} - \Delta y_2^\circ \right)^2 \right] \times \left| \frac{\Delta y_1^\circ + \Delta y_2}{\Delta y_1 + \Delta y_2} \right|^{-2/(\Gamma-1)} \tag{46}$$

Equation (45) is a solution to equation (18). It has been found without the use of confining boundaries, and is similar to that for a free particle in that

the energy ϵ is arbitrary. However, the dynamics of the system restricts the wave function entirely to the interior of I. The wave function oscillates in time according to equation (25), but the probability density (and thus the time-averaged values of observables) is time-independent.

The main difference between the classical and quantum X–Y models is the following. Memory is defined as the location of a final resting place of a trajectory in the classical model, while in the quantum system it is defined by the probability peak in the neighborhood of the same singular point. It is interesting to note that the observation of the memories in both instances is not instantaneous—the classical system approaches zero velocity exponentially, while the quantum system requires a sufficient number of measurements to obtain an accurate average value.

It must be remembered that equation (45) represents an approximate solution. By its nature, the probability density is not symmetric about the singular point, but further asymmetry is introduced with equations (39) (for instance, everything may be rewritten $1 \leftrightarrow 2$).

The right-hand term in equation (46) that is dominant in determining the location of the probability peak only acts in this way if $\Gamma < 1$. This is exactly the same condition that characterizes the stability of the attractor in the classical model. If $\Gamma > 1$, the sign of the exponent changes, giving a near-zero probability around the singular point, which increases with distance from it. In this case, however, the system actually contains the same amount of information. The $\Gamma - 1$ case in this instance may simply exhibit a memory defined by the location of minimum probability.

Finally, since for most cases $\Gamma < 1$, the approximation used does not seem unreasonable. For locations in the activity space far away from the singular point, the probability densities are made small because of the decreasing exponential factor, which minimizes the distorting effects of the linearization at distances far from the singular point.

5.2. Small-Weights Approximation

The previous section provides a solution that, as is often the case, is restricted to symmetric connection weights. Once more using the further approximation

$$F_i = -y_i + y_i^* \tag{47}$$

allows analysis of a small range of nonsymmetric weights. This provides exactly the correct singular point, and for small connection weights distorts the wave function (and probability density) only slightly. How little depends on the size of error considered desirable. Here it is taken to mean approximately that $|w_{ij}| < 0.1$. Equation (32) becomes

$$\frac{dy_1}{-y_1 + y_1^*} = \frac{dy_2}{-y_2 + y_2^*} = \frac{d\psi}{\gamma\psi} \tag{48}$$

The method used in the previous section applied to the left-hand equation yields

$$C_1 = \left|\frac{\Delta y_2}{\Delta y_1}\right| \tag{49}$$

From equation (29) it also follows that

$$-\gamma \frac{dy_1}{y_1 - y_1^*} = \frac{d\psi}{\psi} \tag{50}$$

from which one obtains

$$C_2 = \psi^{-1}|\Delta y_1|^{-\gamma} \tag{51}$$

The solution to equation (18) is then of the form

$$\Phi\left(\left|\frac{\Delta y_2}{\Delta y_1}\right|, \psi^{-1}|\Delta y_1|^{-\gamma}\right) = 0 \tag{52}$$

which, as in the previous section, is specified by the base curve and initial Gaussian. Using the base curve in equation (39) gives

$$C_1 = \left|\frac{\Delta y_2}{\Delta y_1^\circ}\right| \tag{53}$$

Solving for y_2, one obtains

$$y_2 = C_1|\Delta y_1^\circ| + y_2^* \qquad (y_2 > y_2^*) \tag{54}$$

or

$$y_2 = -C_1|\Delta y_1^\circ| + y_2^* \qquad (y_2 < y_2^*) \tag{55}$$

Substitution of these equations into equation (38) gives

$$\psi^\circ = A \exp\left\{-\frac{1}{4\sigma^2}[\pm C_1|\Delta y_1^\circ| - \Delta y_2^\circ]^2\right\}; \qquad (+): y_2 > y_2^*, \quad (-): y_2 < y_2^* \tag{56}$$

Use of equations (51) and (39) produces

$$\psi = \psi^\circ \frac{|\Delta y_1^\circ|^\gamma}{|\Delta y_1|^\gamma} \tag{57}$$

and by putting equation (56) into equation (57) one obtains

$$\psi = A \exp\left\{ -\frac{1}{4\sigma^2} [\pm C_1 |\Delta y_1^\circ| - y_2^\circ]^2 \right\} \frac{|\Delta y_1^\circ|^\gamma}{|\Delta y_1|^\gamma},$$

$$(+): y_2 > y_2^*, \quad (-): y_2 < y_2^* \tag{58}$$

Finally, elimination of C_1 gives a solution for the entire activity space

$$\psi = A \exp\left[-\frac{1}{4\sigma^2} \left(\frac{\Delta y_2 |\Delta y_1^\circ|}{|\Delta y_1|} - \Delta y_2^\circ \right)^2 \right] \frac{|\Delta y_1^\circ|^\gamma}{|\Delta y_1|^\gamma} \tag{59}$$

and the probability density is

$$|\psi|^2 = \psi\psi^* = A^2 \exp\left[-\frac{1}{2\sigma^2} \left(\frac{\Delta y_2 |\Delta y_1^\circ|}{|\Delta y_1|} - \Delta y_2^\circ \right)^2 \right] \left| \frac{\Delta y_1^\circ}{\Delta y_1} \right|^2 \tag{60}$$

This probability density produces the same characteristics as the solution of the previous section, except that the peaking of the probability is even less symmetric, due to the asymmetric solution procedure. Also, the $\Gamma - 1$ term in the exponent has disappeared, forcing one to ensure that this approximation is only applied to systems with "peaks" in the probability density, just as the classical case only allows this approximation to qualitatively describe stable systems (see Section 3). However, this should not be a problem, because for small weights, $\Gamma < 1$ for all cases.

The advantage of this approximation is that it allows a small range of nonsymmetric connection weights to be described. Additionally, it appears simple enough to proceed with a formal normalization procedure.

5.3 Normalization of the Small-Weights Model

To find an explicit expression for A in equation (59), the condition that total probability sums to unity must be used. That is,

$$|\Psi^2| = \int_0^1 \int_0^1 \psi\psi^* \, dy_1 \, dy_2 = 1 \tag{61}$$

must be solved for A in full. Equation (61) is

$$I \equiv \int_0^1 \int_0^1 dy_1 \, dy_2 \, A^2 \left| \frac{\Delta y_1^\circ}{y_1 - y_1^\circ} \right|^2 \exp\left\{ \frac{-1}{2\sigma^2} \left[(y_2 - y_2^*) \left| \frac{\Delta y_1^\circ}{y_1 - y_1^*} \right| - \Delta y_2^\circ \right]^2 \right\} \tag{62}$$

Since it has already been presumed that σ is very small, an approximate result for equation (62) is most easily obtained by treating the "sharp" Gaussian term as a Dirac δ-function. That is (Korn and Korn, 1961, p. 878)

$$\lim_{\sigma \to 0} \frac{1}{\sqrt{\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x)^2\right) = \delta\left\{\frac{(x)}{\sqrt{2}}\right\} \tag{63}$$

which automatically ensures that Δ_1 and Δ_2 are Hermitian operators in this case and in that of Section 5.2. Using the identity (Gel'Fand and Shilov, 1964)

$$\delta(PQ) = \frac{\delta(P)}{Q} + \frac{\delta(Q)}{P} \tag{64}$$

we can write equation (62)

$$I = A^2\sqrt{2\pi}\sigma \int_0^1 dy_1 \left|\frac{\Delta y_1^\circ}{y_1 - y_1^*}\right| \int_0^1 dy_2\, \delta\left(y_2 - y_2^* - \Delta y_2^\circ \left|\frac{y_1 - y_1^*}{\Delta y_1^\circ}\right|\right) \tag{65}$$

where the limit $\sigma \to 0$ is henceforth understood. For the integral over y_2 to be nonvanishing, the condition

$$0 < y_2 = y_2^* - \Delta y_2^\circ \left|\frac{y_1 - y_1^*}{\Delta y_1^\circ}\right| < 1 \tag{66}$$

must hold. This means that the range of the y_1 terms in the left-hand integrand is now

$$\pm\frac{|\Delta y_1^\circ|}{\Delta y_2^\circ}(y_2^* - 1) + y_1^* \lesseqgtr y_1 \lesseqgtr \pm\frac{|\Delta y_1^\circ|}{\Delta y_2^\circ} y_2^* + y_1^* \tag{67}$$

where the top signs correspond to the case where $\Delta y_1 > 0$, and the bottom signs correspond to the case for which $\Delta y_1 < 0$. The simplest case occurs (for $\Delta y_1 > 0$) if

$$y_1^* + \frac{|\Delta y_1^\circ|}{\Delta y_2^\circ}(y_2^* - 1) < 0 \tag{68}$$

and

$$\frac{|\Delta y_1^\circ|}{\Delta y_2^\circ} y_2^* + y_1^* > 1 \tag{69}$$

which leaves the limits of integration over y_1 as originally specified. Since the purpose here is to derive the normalization condition analytically, this condition is presumed from here on. Then integration over y_2 in equation (65) gives

$$I = A^2\sqrt{2\pi}\sigma \int_0^1 dy_1 \left|\frac{\Delta y_1^\circ}{y_1 - y_1^*}\right| \tag{70}$$

The integral over y_1 has a singularity at $y_1 = y_1^*$. This is dealt with by means of a limiting cutoff η around the singular point, resulting in

$$I = \sigma A^2\sqrt{2\pi}|\Delta y_1^\circ| \ln\left(\frac{y_1^*(1 - y_1^*)}{\eta^2}\right) \tag{71}$$

Finally, since $I = 1$,

$$A = \left[(\sigma\sqrt{2\pi}|\Delta y_1^\circ|) \ln\left(\frac{y_1^*(1 - y_1^*)}{\eta^2}\right)\right]^{-1/2} \tag{72}$$

From here on, η is also understood to be evaluated in the limit approaching zero.

Since $\sigma \to 0$ is an approximation used only for mathematical convenience, it is presumed that the limit $\eta \to 0$ takes precedence. This results in A being a vanishingly small positive quantity—a result of normalizing probability in the neighborhood of the singularity.

A useful example illustrating the procedure used in the analysis of the model is to see if the expectation values resulting from this normalization procedure agree reasonably well with the probability peak [equation (60)]. The simplest case is that of (y_2), which is given by

$$\langle y_2\rangle = \int_0^1 \int_0^1 \psi y_2 \psi^* \, dy_1 \, dy_2 \tag{73}$$

Using the same approximations and conditions as above, one obtains

$$\langle y_2\rangle = A^2\sqrt{2\pi}\sigma|\Delta y_1^\circ|y_2^*\left[\ln\left(\frac{y_1^*(1 - y_1^*)}{\eta^2}\right) + \Delta y_2^\circ\right] \tag{74}$$

Equations (72) and (74) yield

$$\langle y_2\rangle = y_2^* + A^2\sqrt{2\pi}\sigma\Delta y_2^\circ \tag{75}$$

and since the right-hand term vanishes in the limit $\eta \to 0$, we have

$$\langle y_2\rangle = y_2^* \tag{76}$$

Next, consider the case of $\langle y_1\rangle$ given by

$$\langle y_1\rangle = \int_0^1 \int_0^1 \psi y_1 \psi^* dy_1 \, dy_2 \tag{77}$$

The same approximations and conditions lead to

$$\langle y_1 \rangle = A^2 \sqrt{2\pi}\sigma \int_0^1 y_1 \frac{\Delta y_1^\circ}{\Delta y_1} dy_1 \tag{78}$$

Using the identity (Korn and Korn; 1961, p. 928)

$$\int dx \frac{x}{ax + b} = \frac{x}{a} - \frac{b}{a^2} \ln(ax + b) \tag{79}$$

integration of equation (78) gives

$$\langle y_1 \rangle = A^2 \sqrt{2\pi} |\Delta y_1^\circ| \sigma \left[1 - 2y_1^* + y_1^* \ln\left(\frac{(1 - y_1^*) y_1^*}{\eta^2} \right) \right] \tag{80}$$

Equation (72) again gives

$$\langle y_1 \rangle = y_1^* \tag{81}$$

in the limit $\eta \to 0$. The expectation values of the activities are therefore the singular point values in the above limit.

As a further check, the expectation value of the Hamiltonian is calculated. If this is written using equation (31), then

$$H = -i\alpha F_1 \frac{\partial}{\partial y_1} - i\alpha F_2 \frac{\partial}{\partial y_2} + i\alpha \tag{82}$$

Then, operating with equation (82) on equation (59) produces

$$H\psi = \epsilon\psi \tag{83}$$

which is exactly equation (24). This means that

$$\psi^* H \psi = \epsilon |\psi|^2 \tag{84}$$

Integrating over all space, one obtains

$$\langle H \rangle = \epsilon \tag{85}$$

as expected.

6. SUMMARY

The approximate solutions to the QNN model discussed above indicate that the network is capable of storing a measurable memory in a manner different than, but equivalent to that of the classical system. This memory is connected to the "wavefunction" by the probability distribution $|\psi|^2$ which is peaked at the singularity (in **F**) in the Schrödinger equation. In full analogy

with the classical system, this singularity occurs at the same location in the activity space as the resting point of the classical dynamical trajectory, and is analyzed using the same linearization procedure. Further, the condition that ensures that the probability is peaked at this point is exactly the same as the classical condition that defines a stable attractor, hence ensuring reliability in both systems.

An additional similarity is that a certain measurement time is required to gain a precise value for the location of the attractor. In the classical system this consists of waiting for the motion of the pseudo "particle" to exponentially slow to the point where it can reasonably be considered to have stopped. Likewise, the quantum system requires a sufficient amount of time to perform enough measurements to determine the average probability distribution.

A couple of final practical points should be mentioned. First, it must be remembered that the wave function is only approximately correct—the initial Gaussian is presumed to be very sharp for mathematical convenience. Second, the equivalent of Planck's constant α remains arbitrary in the model and thus retains this alternate label instead of the customary $\hbar$. As well, as in Feynman's model there is no specific suggestion as to how one might physically implement the activity operators and their connections. However, despite the current lack of available physical experiment, the system is in the meanwhile generally amenable to computer modeling, as well as being a useful starting point for understanding physical systems where $\alpha \to \hbar$.

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